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ILLUSTRATIVE EXAMPLES OF CENTROSYMMETRIC AND NON-CENTROSYMMETRIC ANISOTROPIC FRICTION

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Abstract—Constitutive equations of centrosymmetric and non-centrosymmetric anisotropic friction defined in the preceding companion paper are completed with illustrative examples. Figures illustrate different frictional anisotropy classes and the influence of anisotropic friction on a material point motion.

1. INTRODUCTION

The paper by Zmitrowicz (1992) formulated constitutive equations for centrosymmetric and non-centrosymmetric anisotropic friction. The general constitutive equation of anisotropic friction has the following form

$$\mathbf{t} = -N\{[\mathbf{C}_{10}+\mathbf{C}_{11}\cos(n_1\alpha_v)+\mathbf{C}_{12}\sin(m_1\alpha_v)]\mathbf{v}+\cdots+[\mathbf{C}_{n0}+\mathbf{C}_{n1}\cos(n_n\alpha_v)+\mathbf{C}_{n2}\sin(m_n\alpha_v)]\cdot\underbrace{(\mathbf{v}\otimes\ldots\otimes\mathbf{v})}_{2n-1\text{ copies}}\}, \quad (1)$$

where, $\mathbf{t}$ is the friction force vector, $\mathbf{v}$ is the sliding velocity unit vector, $\mathbf{C}_{ik}(i = 1,2,\ldots,n;\ k = 0,1,2)$ are constant friction tensors of the order $2i$, n_i and m_i are parameters of the trigonometrical functions. Here, we will analyse different simplified forms of eqn (1).

Definitions of anisotropic friction characteristic quantities, i.e. the inclination angle (β), the friction coefficients (μ_α, $\mu_\alpha^\perp$), principal, neutral and extremal value directions, the friction force hodograph, symmetries (rotations, mirror reflections) and anisotropic friction types (isotropic, orthotropic, tetrogonal anisotropic) are given in Zmitrowicz (1992). The investigations reported here are a continuation of the previous paper.

In the present work we are concerned with illustrative examples and representations of some properties of the centrosymmetric and non-centrosymmetric friction by diagrams. Furthermore, an analysis of the motion of a material point in a surface with anisotropic friction properties is presented. We use the same notation as in Zmitrowicz (1992) and we refer to that work.

2. CENTROSYMMETRIC ANISOTROPIC FRICTION

Illustrations of linear and non-linear models of anisotropic friction with constant friction tensors are given by Zmitrowicz (1989). Here, they are completed for two particular cases, i.e. axisymmetric and unidirectional anisotropic friction. In these cases the friction equation has the following form:

$$\mathbf{t} = -N\mathbf{C}_{10}\mathbf{v}. \quad (2)$$

We may take any real numbers as the second order friction tensor coefficients, with the restriction that the friction tensor is positive definite. Generally, the following matrix

$$[\mathbf{C}_{1k}] = \begin{bmatrix} C_k^{11} & C_k^{12} \\ C_k^{21} & C_k^{22} \end{bmatrix}, \quad k = 0,1,2 \quad (3)$$

is a representation of the second order friction tensors $\mathbf{C}_{1k}$. As an example we take the friction coefficient values which are typical for friction in metals. We consider the following cases: axisymmetric anisotropic friction ($C_0^{11} = C_0^{22} = 0,13$; $C_0^{12} = -C_0^{21} = 0,03$) and unidirectional anisotropic friction ($C_0^{11} = C_0^{12} = C_0^{21} = C_0^{22} = 0,05$) (Figs 1, 2).

Three examples of centrosymmetric anisotropic friction with friction tensor coefficients depending on the sliding direction are given. Let the friction force equation restricted to two second order tensors take the form

$$\mathbf{t} = -N[\mathbf{C}_{10} + \mathbf{C}_{11}\cos(4\alpha_v)]\mathbf{v}. \tag{4}$$

Using $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ orthotropic ($C_0^{11} = 0,09$; $C_0^{22} = 0,13$; $C_1^{11} = 0,04$; $C_1^{22} = 0,03$) we get friction with two orthogonal principal directions (Fig. 3). The friction force hodograph is not an ellipse. The group of symmetry is typical for orthotropy.

Let us take the following friction force equation:

$$\mathbf{t} = -N[\mathbf{C}_{10} + \mathbf{C}_{11}\cos(2\alpha_v)]\mathbf{v}, \tag{5}$$

where $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ are isotropic ($C_0^{11} = C_0^{22} = 0,1$; $C_1^{11} = C_1^{22} = 0,03$). Then we have friction with an infinite number of principal directions ($\mu_\alpha^\perp$, β equal to zero) (Fig. 4). In this case, there are two mirror reflections with respect to extremal value directions.

We consider the constitutive equation with second and fourth order tensors in the following form:

$$\mathbf{t} = -N\{\mathbf{C}_{10}\mathbf{v} + [\mathbf{C}_{20} + \mathbf{C}_{22}\sin(4\alpha_v)](\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v})\}. \tag{6}$$

Sixteen elements of the representation of the tensor $\mathbf{C}_{2k}$ can be arranged in a table:

$$[\mathbf{C}_{2k}] = \begin{array}{c} \begin{array}{cccc} 11 & 22 & 21 & 12 \end{array} \\ \begin{bmatrix} C_k^{1111} & C_k^{1122} & C_k^{1121} & C_k^{1112} \\ C_k^{2211} & C_k^{2222} & C_k^{2221} & C_k^{2212} \\ C_k^{2111} & C_k^{2122} & C_k^{2121} & C_k^{2112} \\ C_k^{1211} & C_k^{1222} & C_k^{1221} & C_k^{1212} \end{bmatrix} \end{array} \begin{array}{c} \\ 11 \\ 22, \\ 21 \\ 12 \end{array} \quad k = 0, 1, 2. \tag{7}$$

They are ordered by pairs of indices. Using $\mathbf{C}_{10}$ isotropic ($C_0^{11} = C_0^{22} = 0,07$) and $\mathbf{C}_{20}$, $\mathbf{C}_{22}$ tetragonal anisotropic

$$(C_0^{1111} = C_0^{2222} = 2C_2^{1111} = 2C_2^{2222} = 0,02; \quad C_0^{1122} = C_0^{2211} = 2C_2^{1122} = 2C_2^{2211} = 0,08;$$
$$C_0^{2121} = C_0^{1212} = 2C_2^{2121} = 2C_2^{1212} = 0,01; \quad C_0^{2112} = C_0^{1221} = 2C_2^{2112} = 2C_2^{1221} = 0,06)$$

we obtain friction with a four-fold rotation axis (Fig. 5). There are no mirror reflections in this case.

In the examples only the non-zero elements of the tensors are given.

Figures 1–5 show the following plots: (a) the friction coefficient $\mu_\alpha^\perp$, (b) the friction coefficient μ_α referring to polar coordinates, (c) the inclination angle β, and (d) the friction force hodograph with respect to an orthogonal reference system. Principal directions of friction (dash–dot line) and zeros of the function $\beta = f(\alpha_v)$ are shown in the figures.

Anisotropic friction in a contact between two bodies changes the nature of the relative motion of the bodies. Here, we analyse properties of a material point motion in a plane with frictional anisotropy. Figures 6–10 present trajectories of the material point motion in the plane with frictional properties shown by Figs 1–5, respectively.

The following equation describes motion

$$m\ddot{\mathbf{q}} = \mathbf{t}, \tag{8}$$

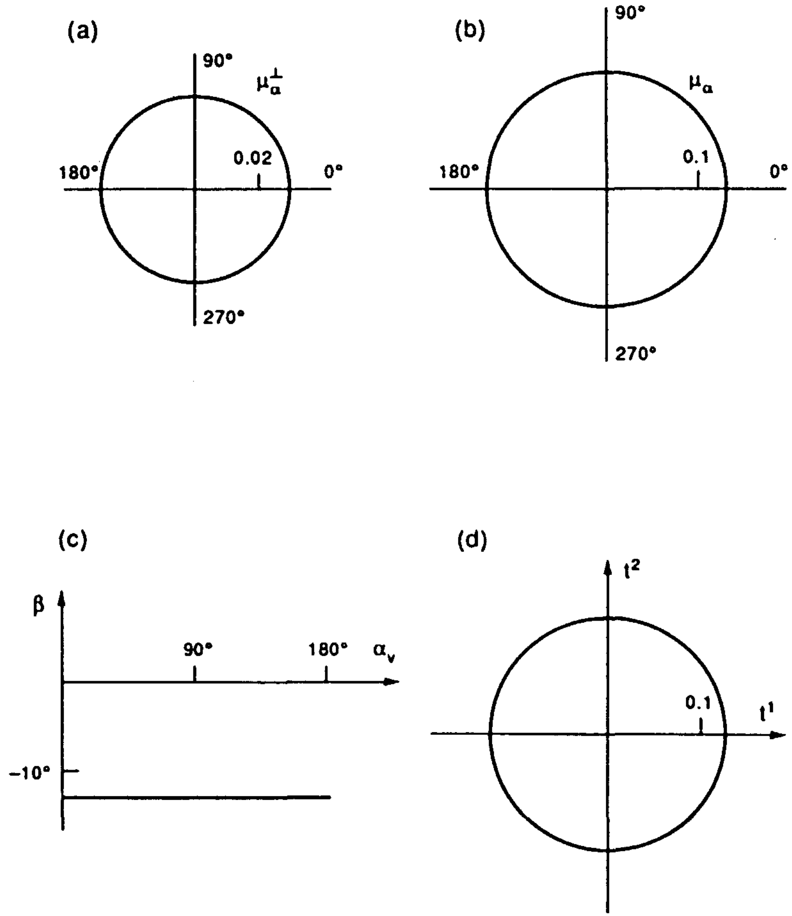


Fig. 1. Illustration of the axisymmetric anisotropic friction: (a) friction coefficient $\mu_s^\perp$; (b) friction coefficient μ_s; (c) inclination angle β; (d) friction force hodograph.

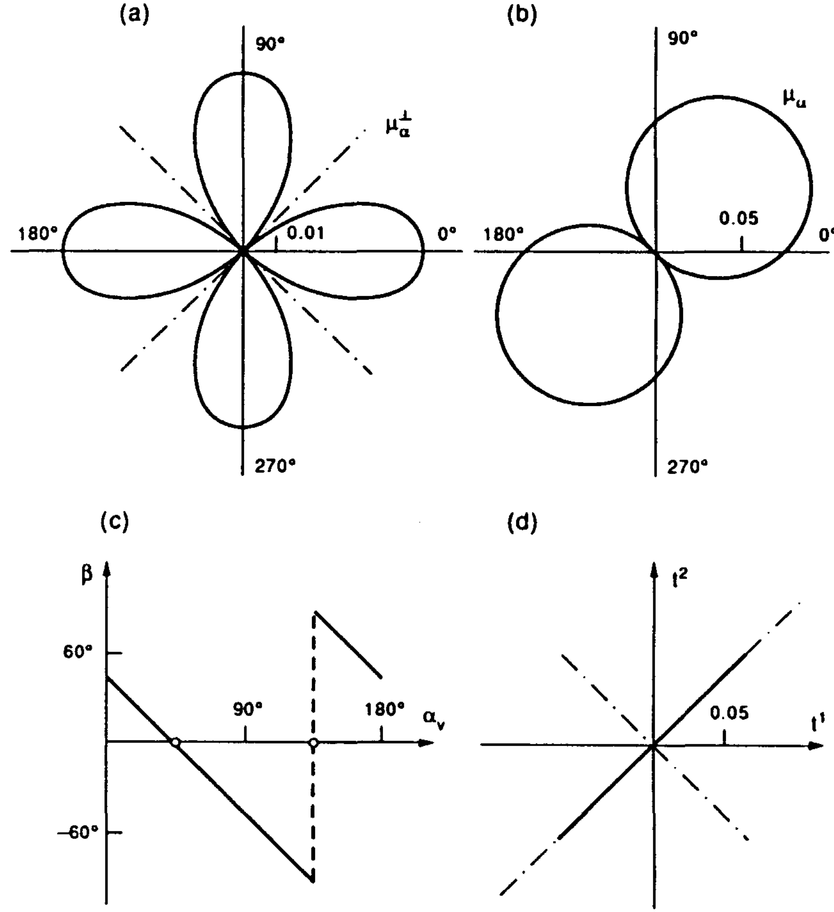


Fig. 2. Illustration of the unidirectional anisotropic friction: (a) friction coefficient $\mu_s^\perp$; (b) friction coefficient μ_s; (c) inclination angle β; (d) friction force hodograph.

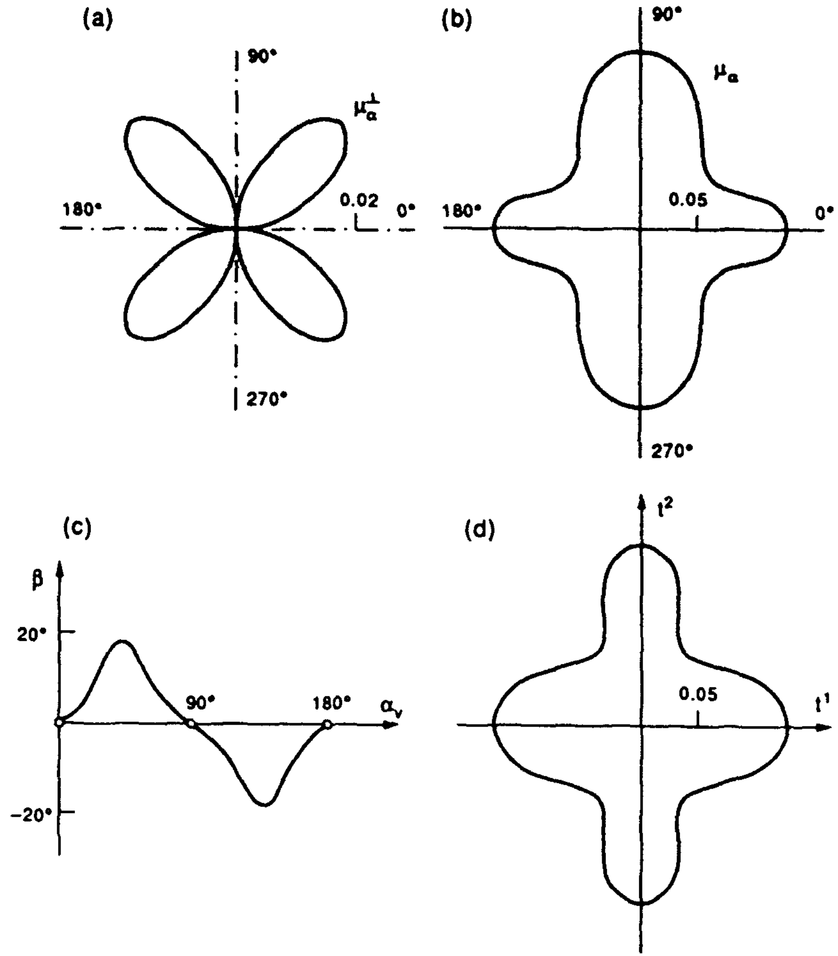


Fig. 3. Illustration of the centrosymmetric anisotropic friction with two orthogonal principal directions: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

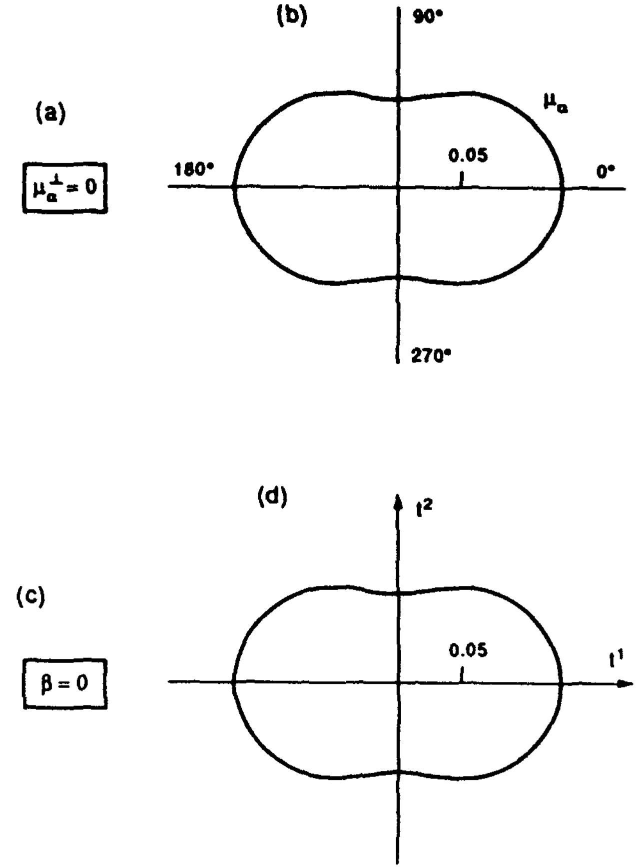


Fig. 4. Illustration of the centrosymmetric anisotropic friction with an infinite number of principal directions: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

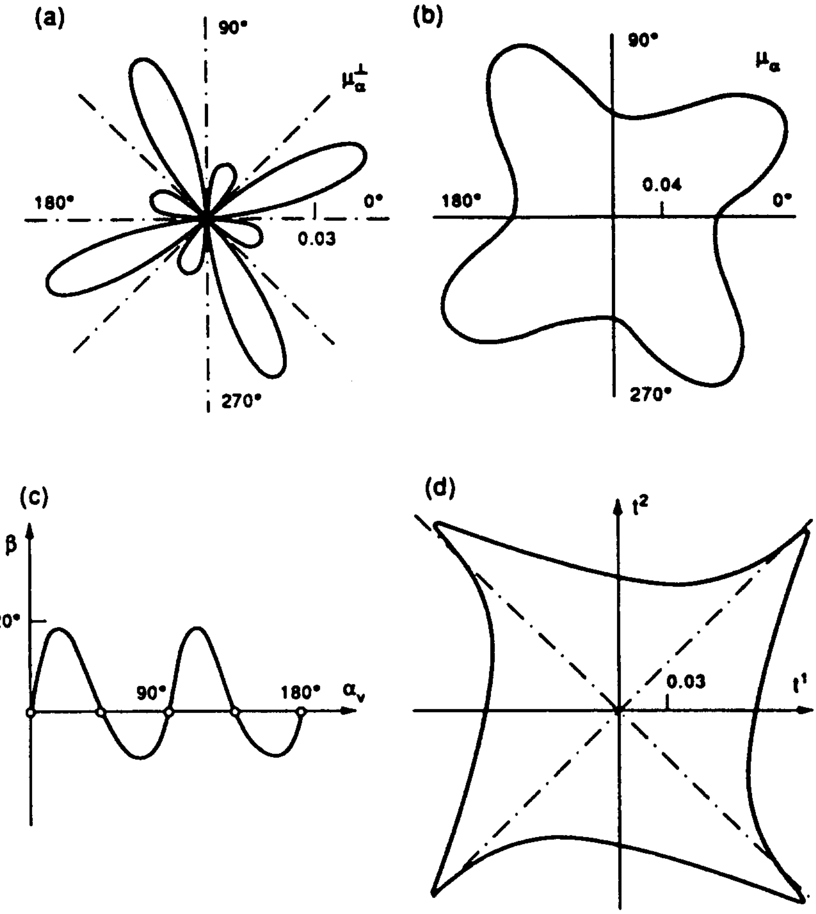


Fig. 5. Illustration of the centrosymmetric anisotropic friction with a four-fold rotation axis: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

where m is the mass of the material point, $\mathbf{q}$ its position vector and $\mathbf{t}$ the friction force vector. The velocity $\dot{\mathbf{q}}_0$ is taken as the initial condition of motion. Different directions of the initial velocity are analysed. They are given by the angle α_0,

$$[\dot{\mathbf{q}}_0] = [\dot{q}_0 \cos\alpha_0 \quad \dot{q}_0 \sin\alpha_0]^{\mathrm{T}} \tag{9}$$

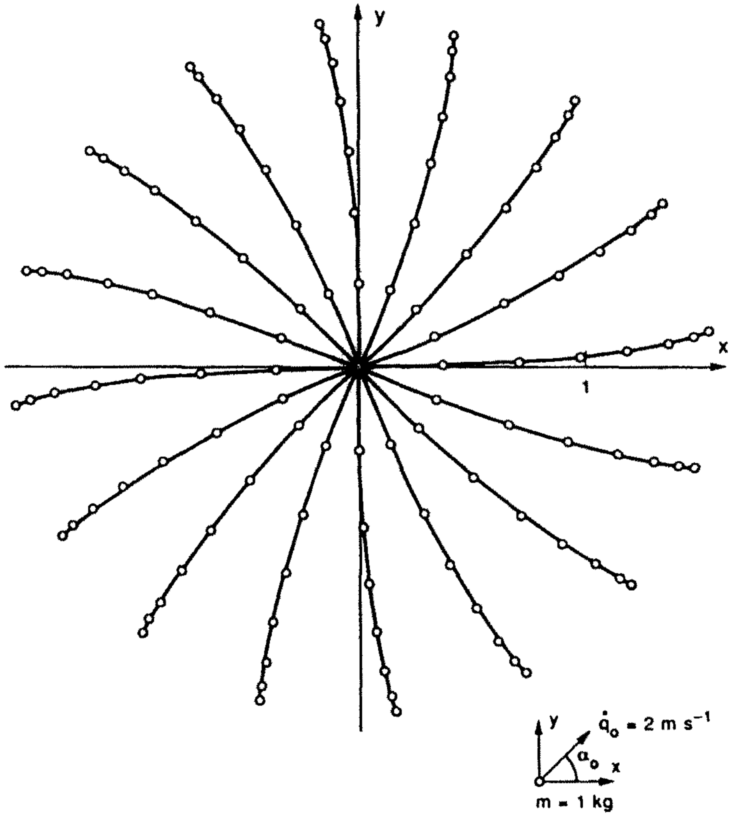
and specified in the figures. The non-linear eqn (8) is solved by means of the Runge–Kutta fourth-order method.

The material point motion is a retarded motion if the friction force acts. The length of the point trajectory depends on the frictional resistance. Intervals between points on the trajectory shown in Figs 6–10 correspond to constant time intervals (0, 2 s). In the case of unidirectional friction (Fig. 7), the point has so much initial energy that it achieves frictionless direction. Next, the point moves uniformly towards infinity.

The trajectory is a segment collinear with the principal direction of friction when the motion occurs in this direction. If all sliding directions are principal (Fig. 9), then all trajectories are segments of lines. The trajectories are curved if a normal component of the friction force acts.

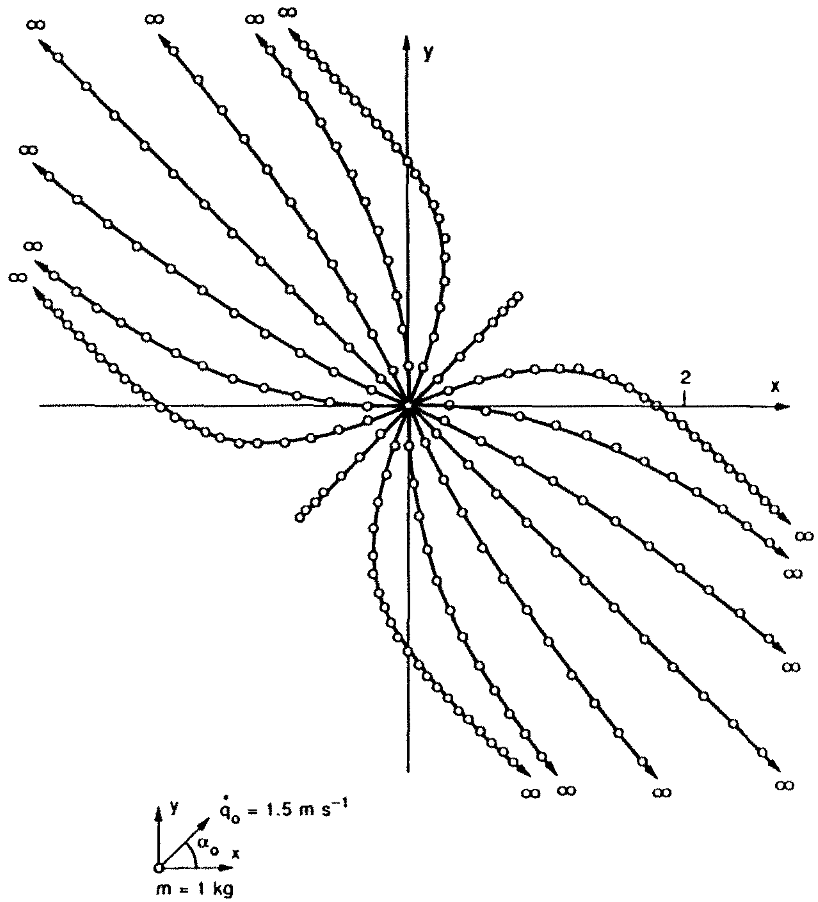
3. NON-CENTROSYMMETRIC ANISOTROPIC FRICTION

We consider examples of non-centrosymmetric anisotropic friction described by friction tensor coefficients depending on the sliding direction. Let the friction force equation



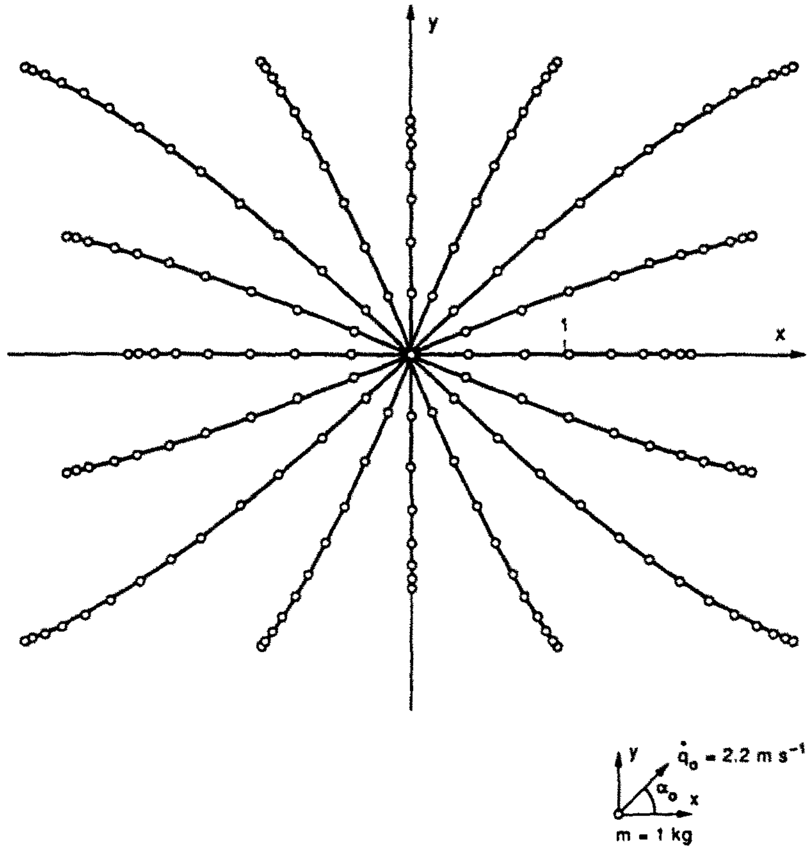
α_0	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 6. Motion of a material point in a plane with axisymmetric anisotropic friction.



α_0	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 7. Motion of a material point in a plane with unidirectional anisotropic friction.



α_0								
	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 8. Motion of a material point in a plane with centrosymmetric anisotropic friction with two orthogonal principal directions.

be given by

$$\mathbf{t} = -N(\mathbf{C}_{10}+\mathbf{C}_{11}\cos\alpha_v)\mathbf{v}. \tag{10}$$

If tensors $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ are anisotropic without principal directions ($C_0^{11} = 2C_1^{11} = 0,12$; $C_0^{12} = 2C_1^{12} = -0,06$; $C_0^{21} = 2C_1^{21} = 0,02$; $C_0^{22} = 2C_1^{22} = 0,14$) then we have anisotropic friction without principal directions and with one neutral direction (Fig. 11). If $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ are anisotropic with one principal direction ($C_0^{11} = 2C_1^{11} = 0,1$; $C_0^{12} = 2C_1^{12} = -0,02$; $C_0^{21} = 2C_1^{21} = 0,02$; $C_0^{22} = 2C_1^{22} = 0,06$) then we get anisotropic friction with one principal direction and one neutral direction (Fig. 12). Using $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ orthotropic ($C_0^{11} = 0,12$; $C_0^{22} = 0,1$; $C_1^{11} = 0,09$; $C_1^{22} = 0,06$) we have friction with two orthogonal and two unidirectional principal directions, and with one neutral direction (Fig. 13).

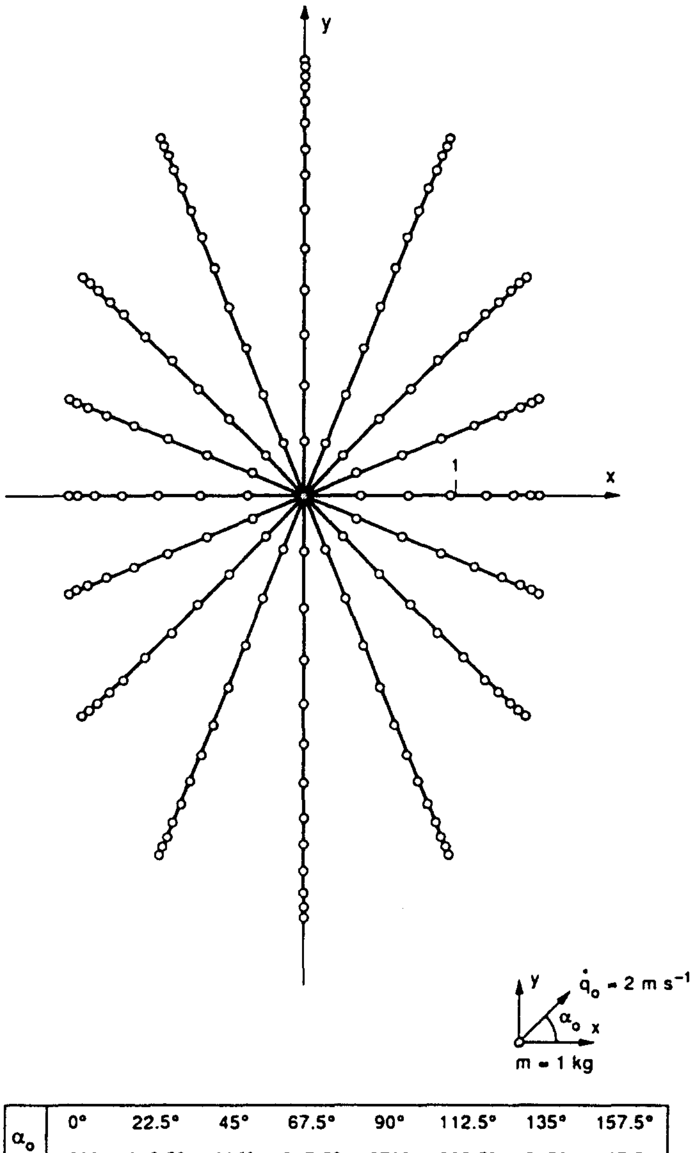
Let us consider the friction force equation in the following form

$$\mathbf{t} = -N[\mathbf{C}_{10}+\mathbf{C}_{11}\cos(3\alpha_v)]\mathbf{v}. \tag{11}$$

If tensors $\mathbf{C}_{10}$, $\mathbf{C}_{11}$ are isotropic ($C_0^{11} = C_0^{22} = 0,1$; $C_1^{11} = C_1^{22} = 0,03$) then eqn (11) defines friction with an infinite number of principal and three neutral directions (Fig. 14).

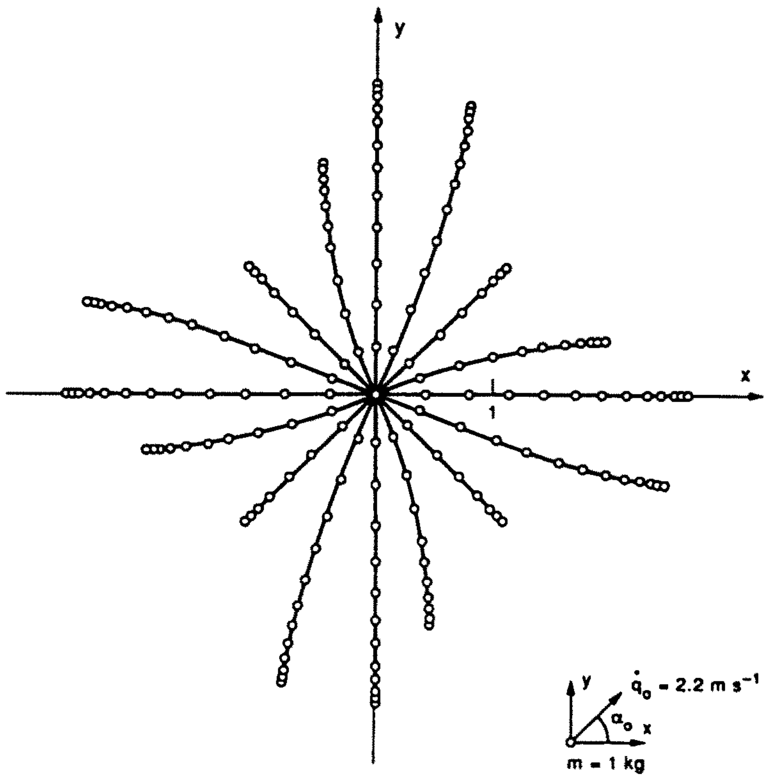
We take the following constitutive equation with fourth order tensors

$$\mathbf{t} = -N[\mathbf{C}_{10}\mathbf{v}+(\mathbf{C}_{20}+\mathbf{C}_{21}\cos\alpha_v)(\mathbf{v}\otimes\mathbf{v}\otimes\mathbf{v})]. \tag{12}$$



α_o	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 9. Motion of a material point in a plane with centrosymmetric anisotropic friction with an infinite number of principal directions.



α_0								
	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 10. Motion of a material point in a plane with centrosymmetric anisotropic friction with a four-fold rotation axis.

Using C_{10} isotropic ($C_0^{11} = C_0^{22} = 0,07$) and C_{20}, C_{21} tetragonal anisotropic ($C_0^{1111} = C_0^{2222} = 2C_1^{1111} = 2C_1^{2222} = 0,02$; $C_0^{1122} = C_0^{2211} = 2C_1^{1122} = 2C_1^{2211} = 0,08$; $C_0^{2121} = C_0^{1212} = 2C_1^{2121} = 2C_1^{1212} = 0,01$; $C_0^{2112} = C_0^{1221} = 2C_1^{2112} = 2C_1^{1221} = 0,06$) we obtain friction with four principal directions and one neutral direction (Fig. 15).

Only the non-zero elements of the tensors are given.

Principal (dash–dot line) and neutral (broken line) directions of friction and zeros of the function $\beta = f(\alpha_0)$ are shown in Figs 11–15.

Figures 16–20 present trajectories of the material point motion in the surface with frictional anisotropies described by models without central symmetry. The frictional properties of the surface are shown in Figs 11–15. The trajectories have finite length and the material point motion is a retarded motion.

4. CONCLUSIONS

(1) Constitutive equations for anisotropic friction enable prediction of the behaviour of a contact that can be confirmed by experimental observations. It might be expected that future investigations will give new experimental data on anisotropic friction. Then an adequate quantitative comparison of experimental and theoretical results will be possible.

(2) The anisotropic friction models have a wide range of applications due to their general character and can be used in a computer simulation of the phenomenon and in practical numerical calculations of contact, rubbing and tribological problems.

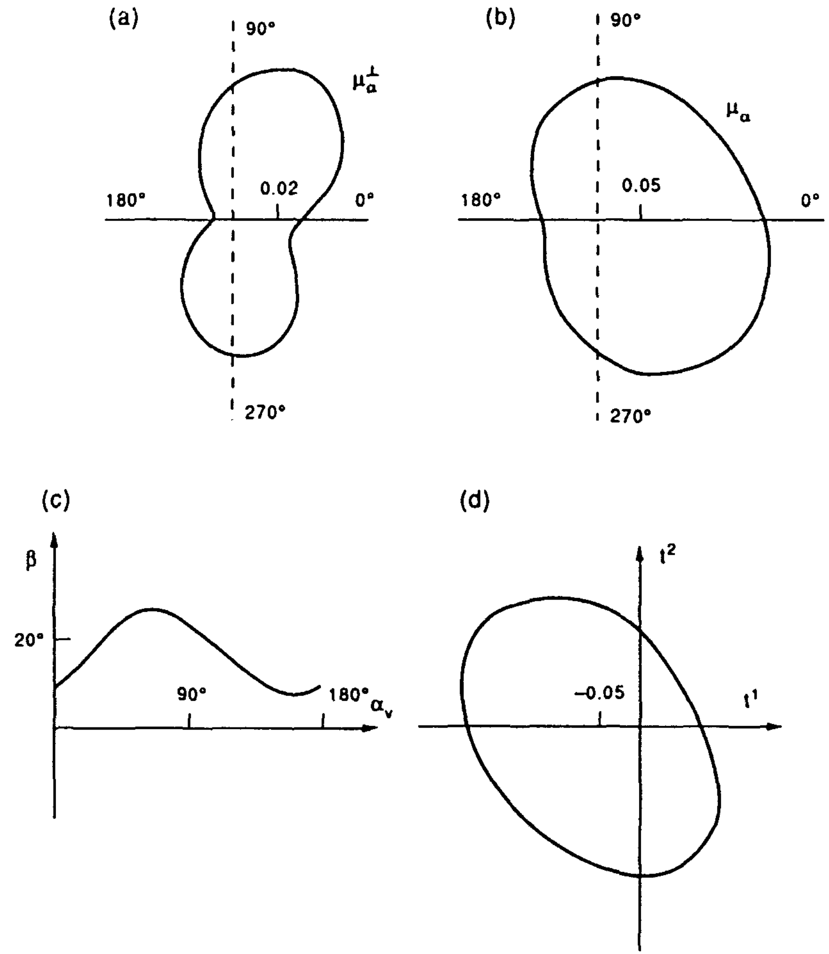


Fig. 11. Illustration of the non-centrosymmetric anisotropic friction without principal directions: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

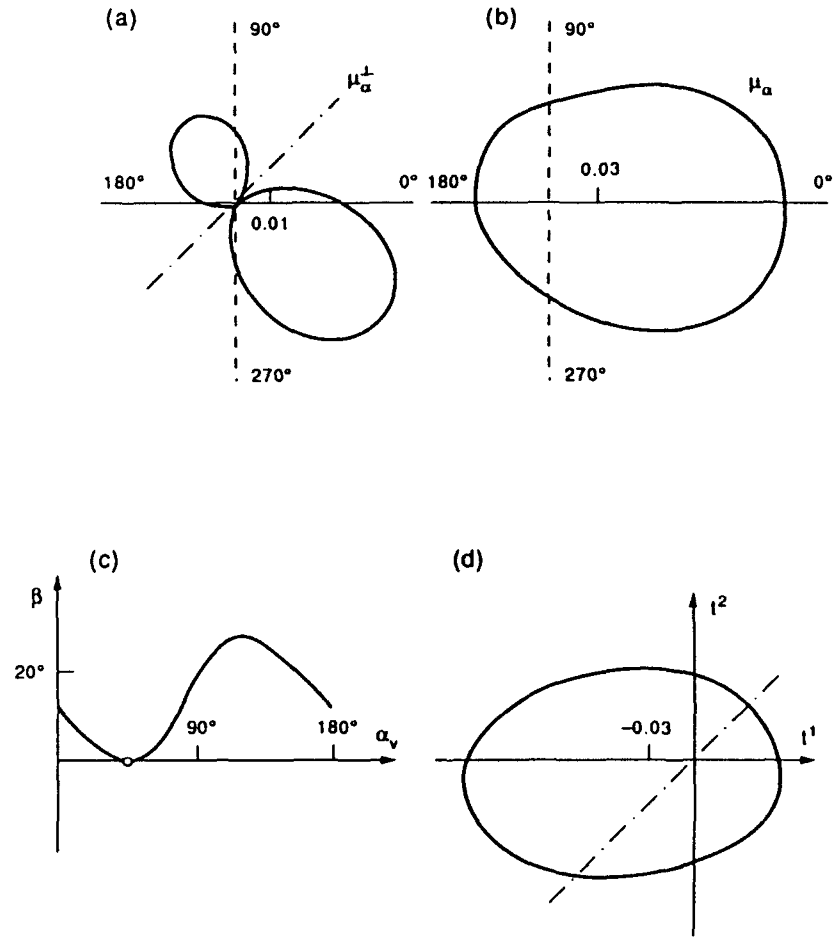


Fig. 12. Illustration of the non-centrosymmetric anisotropic friction with one principal direction: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

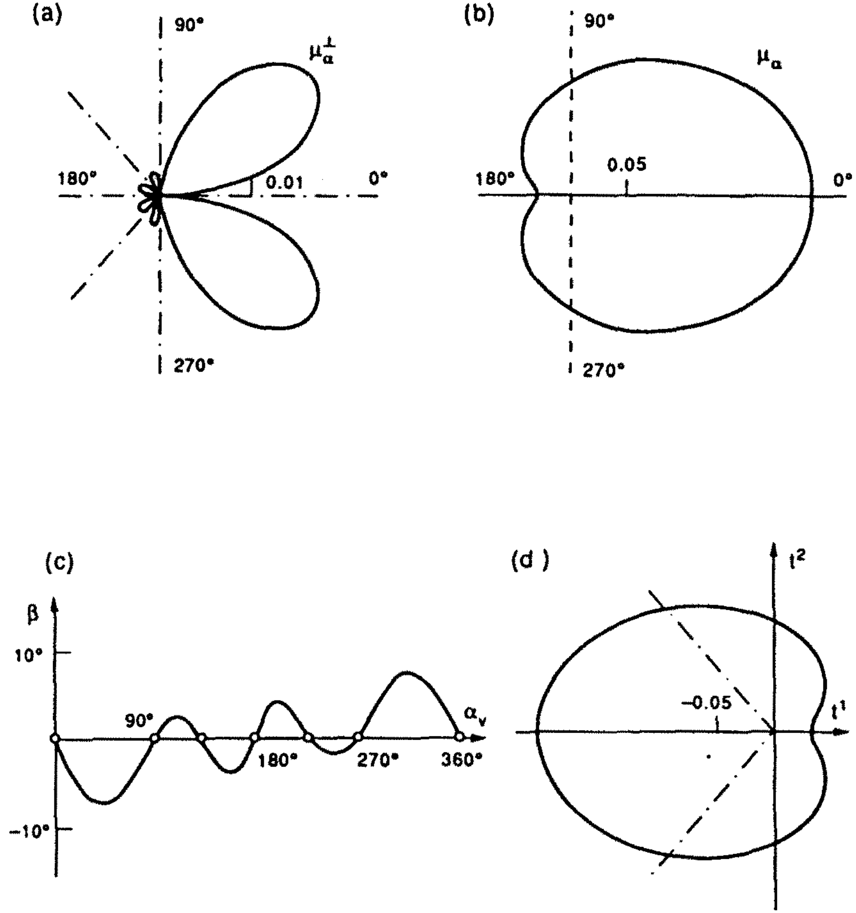


Fig. 13. Illustration of the non-centrosymmetric anisotropic friction with two orthogonal principal directions and two unidirectional principal directions: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

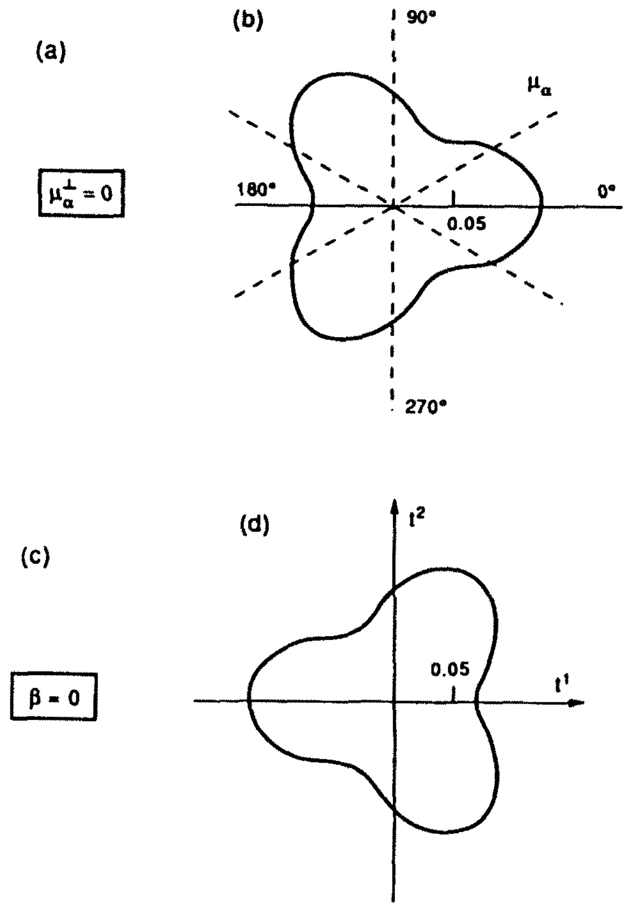


Fig. 14. Illustration of the non-centrosymmetric anisotropic friction with an infinite number of principal directions: (a) friction coefficient $\mu_\alpha^\perp$; (b) friction coefficient μ_α; (c) inclination angle β; (d) friction force hodograph.

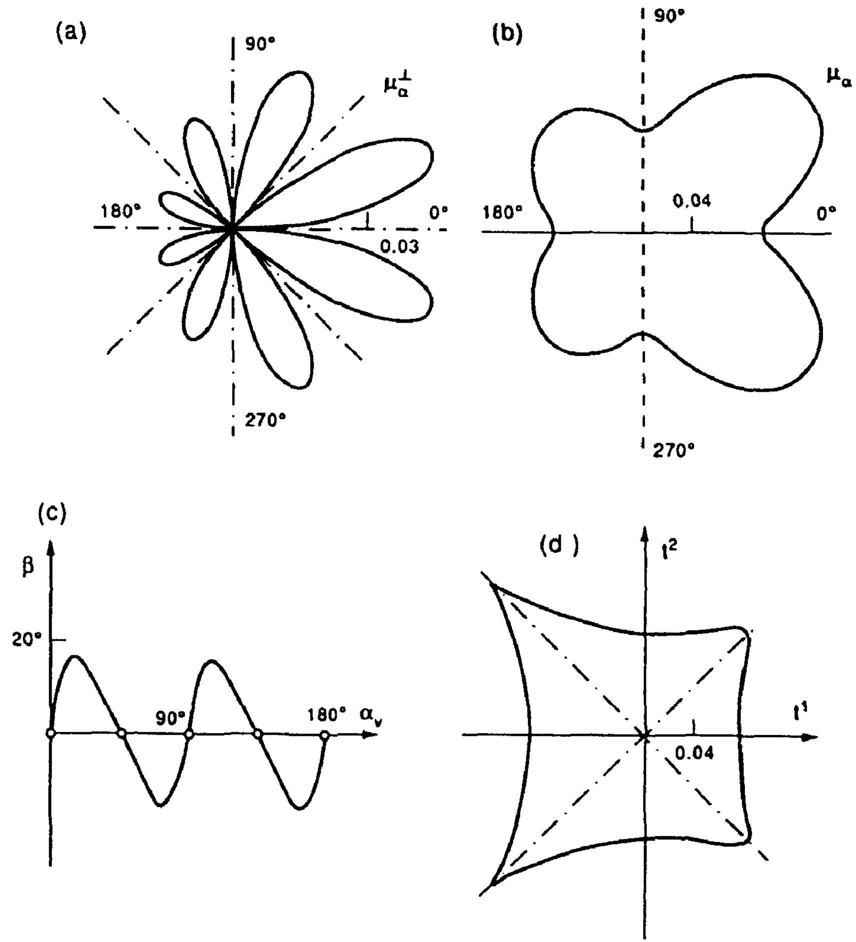
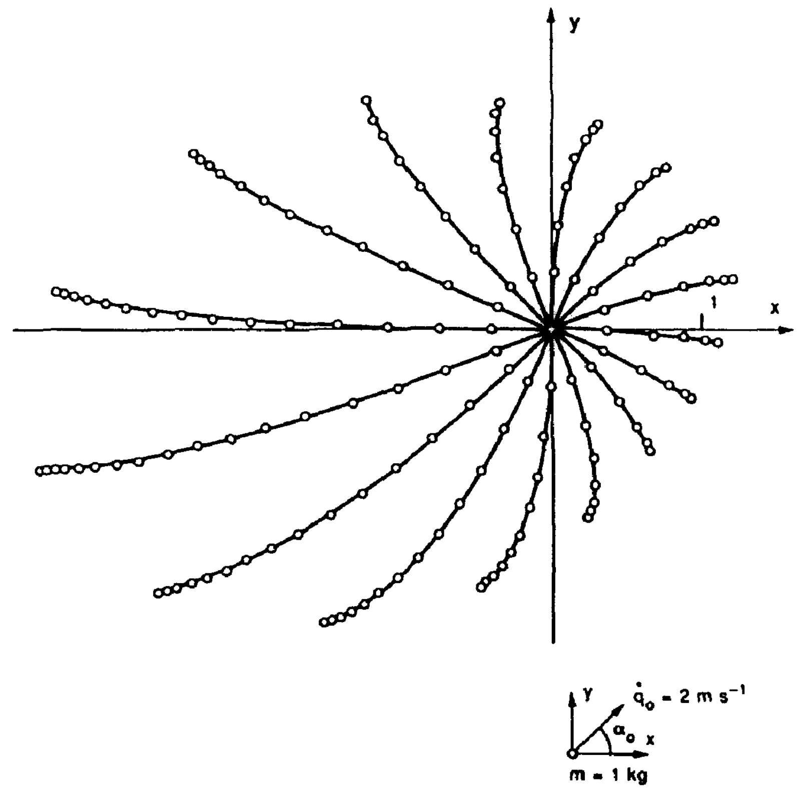
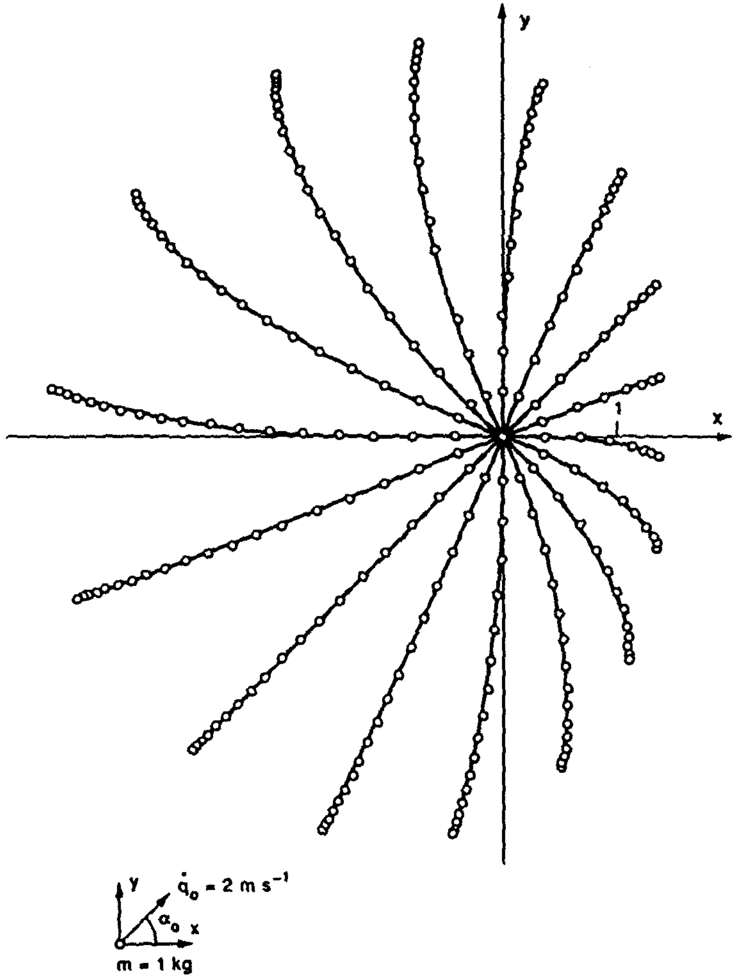


Fig. 15. Illustration of the non-centrosymmetric anisotropic friction with four principal directions: (a) friction coefficient $\mu_s^\perp$; (b) friction coefficient μ_s; (c) inclination angle β; (d) friction force hodograph.



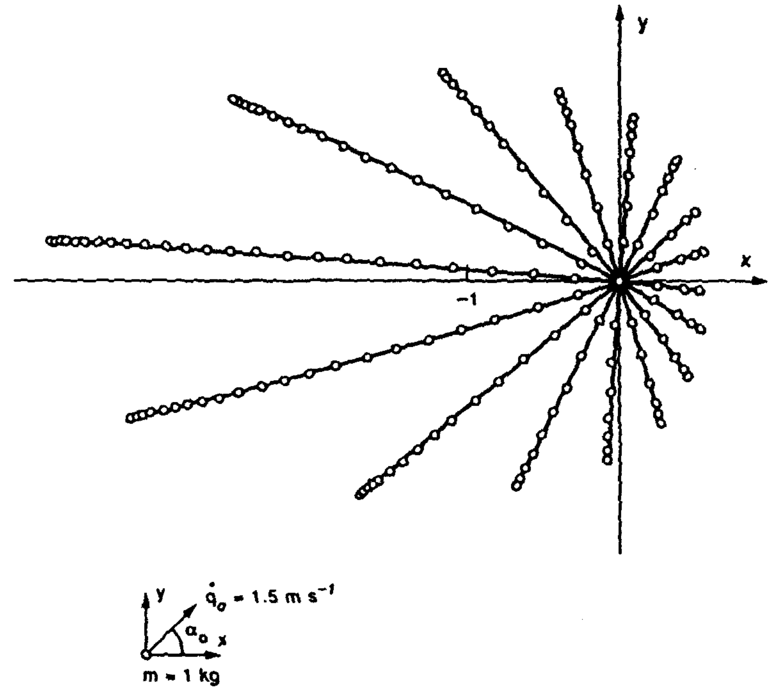
α_0	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 16. Motion of a material point in a plane with non-centrosymmetric anisotropic friction without principal directions.



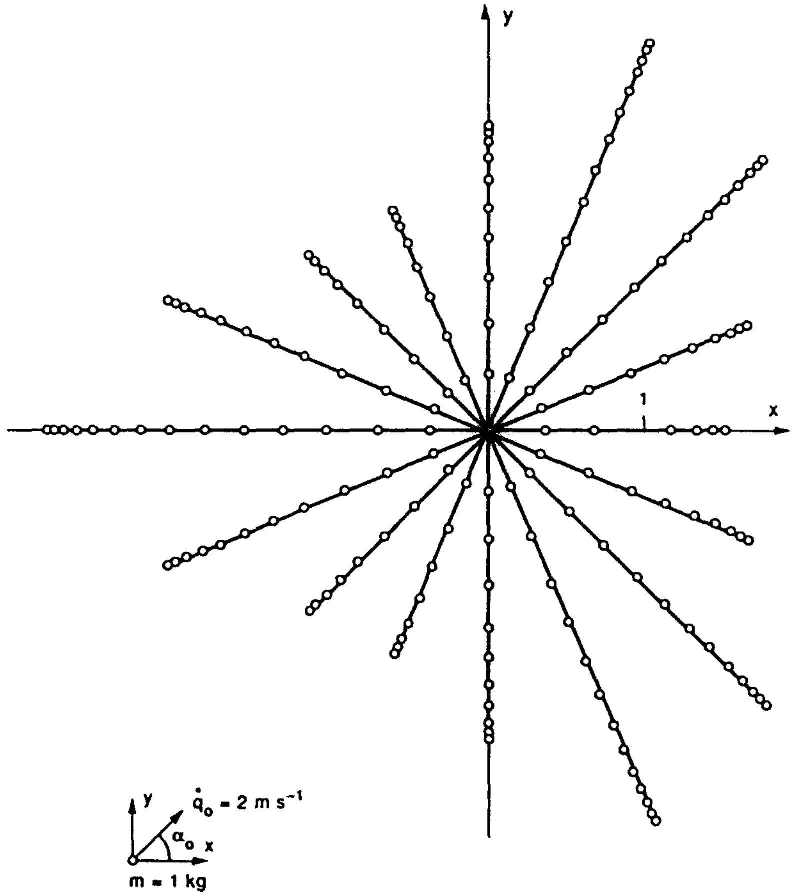
α_o								
	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 17. Motion of a material point in a plane with non-centrosymmetric anisotropic friction with one principal direction.



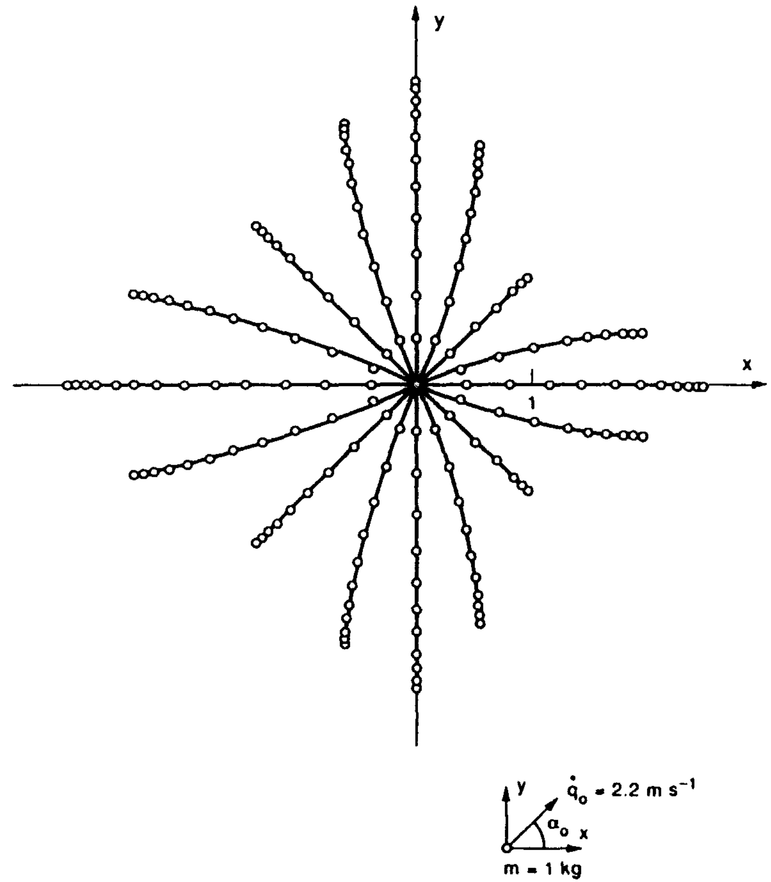
α_o								
	17.5°	40°	62.5°	85°	107.5°	130°	152.5°	175°
	197.5°	220°	242.5°	265°	287.5°	310°	332.5°	355°

Fig. 18. Motion of a material point in a plane with non-centrosymmetric anisotropic friction with two orthogonal principal directions and two unidirectional principal directions.



α_0								
	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 19. Motion of a material point in a plane with non-centrosymmetric anisotropic friction with an infinite number of principal directions.



α_0								
	0°	22.5°	45°	67.5°	90°	112.5°	135°	157.5°
	180°	202.5°	225°	247.5°	270°	292.5°	315°	337.5°

Fig. 20. Motion of a material point in a plane with non-centrosymmetric anisotropic friction with four principal directions.

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